DSC-1A DIFFERENTIAL CALCULUS
BS:104

Theory: 4 credits and Practicals: 1 credit
Theory: 4 hours/week and Practicals: 2 hours/week

Objective: the course is aimed at exposing the students to some basic notions in differential calculus.

Outcome: by the time students completes the course they realize wide ranging applications of the subject.

Unit – I

Successive differentiation:
Higher order derivatives, Calculation of the nth derivative, Some standard results, Determination of nth derivative of rational functions, The nth derivatives of the products of the powers of sines and cosines, Leibnitz's theorem, The nth derivative of the product of two functions.
Expansion of Functions:
Maclaurin's theorem, Taylor's theorem.
Mean Value Theorems:
Rolle's theorem, Lagrange's mean value theorem, Meaning of the sign of derivative, Graphs of hyperbolic functions, Cauchy's mean value theorem, Higher derivatives, Formal expansions of functions.

Unit – II

Indeterminate Forms:
Indeterminate forms, The indeterminate form 0/0, The indeterminate form ∞/∞, The indeterminate form 0, ∞, The indeterminate form $\infty - \infty$, The indeterminate forms 0^0, 1^∞, ∞^0.
Curvature and Evolutes:
Introduction, Definition of curvature, Length of arc as a function, Derivative of arc, Radius of curvature-cartesian equations, Newtonian method, Centre of curvature, Chord of curvature, Evolutes and involutes, Properties of the evolute.

Unit – III

Partial Differentiation – Homogeneous Functions – Total Derivative:
Introduction, Functions of two variables, Neighbourhood of a point (a, b), Continuity of a Function of two variables, continuity at a point, Limit of a function of two variables, Partial derivatives, Geometrical representation of a function of two variables, Homogeneous functions, Theorem on total differentials; composite functions; differentiation of composite functions; implicit functions.

Unit – IV

Maxima and Minima:
Maxima and minima of function of two variables, Lagrange's method of undetermined multipliers.

Asymptotes:
Definition, Determination of asymptotes, Working rules of determining asymptotes, Asymptotes by inspection, Intersection of a curve and its asymptotes, Asymptotes by expansion, Position of a curve with respect to an asymptote, Asymptotes in polar co-ordinates.
Envelopes:
One parameter family of curves, Consider the family of straight lines, Definition, Determination of envelope, Theorem, To prove that, in general, the envelope of a family of curves touches each member of the family, If A, B, C are functions of x and y and m is a parameter then the envelope of $Am^2+Bm+C = 0$ is $B^2 = 4AC$, Two parameters connected by a relation, When the equation to a family of curves is not given, but the law is given in accordance with which any member of the family can be determined, Envelopes of polar curves, Envelopes of normals(Evolutes).

Text: Shanti Narayan and Mittal, Differential Calculus

References: William Anthony Granville, Percey F Smith and William Raymond Longley, Elements of the Differential and integral calculus

Joseph Edwards, Differential calculus for beginners

Smith and Minton, Calculus

Elis Pine, How to Enjoy Calculus

Hari Kishan, Differential Calculus

2.1.1 Practicals Question Bank

Differential Calculus

Unit-I

1. If $u = \tan^{-1} x$ prove that
$$(1+x^2)\frac{d^2u}{dx^2} + 2x\frac{du}{dx} = 0$$
and hence determine the values of the derivatives of u when $x = 0$.

2. If $y = \sin(m \sin^{-1} x)$ show that
$$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$$
and find $y_n(0)$

3. If U_n denotes the nth derivative of $\frac{Lx+M}{x^2-2Bx+C}$., prove
$$\frac{x^2-2Bx+C}{(n+1)(n+2)}U_{n+2} + \frac{2(x-B)}{n+1}U_{n+1} + U_n = 0$$

4. If $y = x^2 e^x$, then
$$\frac{d^n y}{dx^n} = \frac{1}{2}n(n-1)\frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)y.$$

5. Determine the intervals in which the function
$$(x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x}$$
is increasing or decreasing.

6. Separate the intervals in which the function
$$\frac{(x^2+x+1)}{(x^2-x+1)}$$
is increasing or decreasing.

7. Show that if $x > 0$,

   (i) $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$.

   (ii) $x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$.

8. Prove that
$$e^{ax}\sin bx = bx + abx^2 + \frac{3a^2b - b^3}{3!}x^3 + \ldots\ldots + \frac{(a^2+b^2)^{\frac{1}{2}n}}{n!}x^n \sin(n\tan^{-1}\frac{b}{a}) + \ldots..$$

9. Show that
$$\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6\ldots\ldots\ldots$$

10. Show that

$$e^{m\tan^{-1}x} = 1 + mx + \frac{m^2}{2!}x^2 + \frac{m(m^2-2)}{3!}x^3 + \frac{m^2(m^2-8)}{4!}x^4 + ...$$

Unit-II

11. Find the radius of curvature at any point on the curves

(i) $y = c\cosh(\frac{x}{c})$. (Catenary)

(ii) $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$.

(iii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (Astroid)

(iv) $x = \frac{(a\cos t)}{t}, y = \frac{(a\sin t)}{t}$.

12. Show that for the curve

$$x = a\cos\theta(1+\sin\theta), y = a\sin\theta(1+\cos\theta),$$

the radius of curvature is a at the point for which the value of the parameter is $\frac{-\pi}{4}$.

13. Prove that the radius of curvature at the point $(-2a, 2a)$ on the curve $x^2y = a(x^2+y^2)$ is $-2a$.

14. Show that the radii of curvature of the curve

$$x = ae^{\theta}(\sin\theta - \cos\theta), y = ae^{\theta}(\sin\theta + \cos\theta)$$

and its evolute at corresponding points are equal.

15. Show that the whole length of the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is $4(\frac{a^2}{b} - \frac{b^2}{a})$.

16. Show that the whole length of the evolute of the astroid

$$x = a\cos^3\theta, y = a\sin^3\theta$$

is $12a$

17. Evaluate the following:

(i) $\lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2}$

(ii) $\lim_{x\to 0}\frac{x\cos x - \log(1+x)}{x^2}$

(iii) $\lim_{x\to 0}\frac{e^x\sin x - x - x^2}{x^2 + x\log(1-x)}$

(iv) $\lim_{x\to 0}\{\frac{1}{x} - \frac{1}{x^2}\log(1+x)\}$

18. If the limit of

$$\frac{\sin 2x + a\sin x}{x^8}$$

as x tends to zero, be finite, find the value of a and the limit.

19. Determine the limits of the following functions:

(i) $x \log(\tan x), (x \to 0)$ *(ii)* $x \tan(\pi/2 - x), (x \to 0)$

(iii) $(a - x)\tan(\pi x/2a), (x \to 0)$

20. Determine the limits of the following functions:

(i) $\frac{e^x - e^{-x} - x}{x^2 \sin x}, (x \to 0)$ *(ii)* $\frac{\log x}{x^3}, (x \to \infty)$

(iii) $\frac{1 + x\cos x - \cosh x - \log(1+x)}{\tan x - x}, (x \to 0)$ *(iv)* $\frac{\log(1+x)\log(1-x) - \log(1-x^2)}{x^4}, (x \to 0)$

Unit-III

21. If $z = xyf(x/y)$ then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

22. If $z(x+y) = x^2 + y^2$ then show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

23. If $z = 3xy - y^3 + (y^2 - 2x)^{\frac{3}{2}}$, verify that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad \textit{and} \quad \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

24. If $z = f(x + ay) + \varphi(x - ay)$, prove that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

25. If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, find

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}.$$

26. If $f(x, y) = 0$, $\varphi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y}.\frac{\partial \varphi}{\partial z}.\frac{dz}{dx} = \frac{\partial f}{\partial x}.\frac{\partial \varphi}{\partial y}.$$

27. If $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a$, show that

$$\frac{d^2 y}{dx^2} = \frac{a}{(1 - x^2)^{\frac{3}{2}}}.$$

28. Given that $f(x, y) \equiv x^3 + y^3 - 3axy = 0$, show that

$$\frac{d^2 y}{dx^2}.\frac{d^2 x}{dy^2} = \frac{4a^6}{xy(xy - 2a^2)^3}.$$

29. If u and v are functions of x and y defined by

$$x = u + \mathrm{e}^{-v} \sin u, y = v + \mathrm{e}^{-v} \cos u,$$

prove that

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

30. If $H = f(y - z, z - x, x - y)$; prove that,

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z} = 0.$$

Unit-IV

31. Find the minimum value of $x^2 + y^2 + z^2$ when

   (i) $x + y + z = 3a$

   (ii) $xy + yz + zx = 3a^2$

   (iii) $xyz = a^3$

32. Find the extreme value of xy when

$$x^2 + xy + y^2 = a^2.$$

33. In a plane triangle, find the maximum value of

$$\cos A \cos B \cos C.$$

34. Find the envelope of the family of semi-cubical parabolas

$$y^2 - (x + a)^3 = 0.$$

35. Find the envelope of the family of ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   where the parameters a,b are connected by the relation

$$a + b = c;$$

   c, being a constant.

36. Show that the envelope of a circle whose centre lies on the parabola $y^2 = 4ax$ and which passes through its vertex is the cissoid

$$y^2(2a + x) + x^3 = 0.$$

37. Find the envelope of the family of the straight lines $\frac{x}{a} + \frac{y}{b} = 1$ where a,b are connected by the relation

(i) $a + b = c$. ***(ii)*** $a^2 + b^2 = c^2$. ***(iii)*** $ab = c^2$.

c is a constant.

38. Find the asymptotes of

$$x^3 + 4x^2y + 4xy^2 + 5x^2 + 15xy + 10y^2 - 2y + 1 = 0.$$

39. Find the asymptotes of

$$y^3 + x^3 + y^2 + x^2 - x + 1 = 0.$$

40. Find the asymptotes of the following curves

(i) $xy(x + y) = a(x^2 - a^2)$

(ii) $y^3 - x^3 + y^2 + x^2 + y - x + 1 = 0$.
